

Five-Dimensional Mechanics as the Starting Point for the Magueijo-Smolín Doubly Special Relativity

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We discuss a way to obtain the doubly special relativity kinematical rules (the deformed energy-momentum relation and the nonlinear Lorentz transformations of momenta) starting from a singular Lagrangian action of a particle with linearly realized $SO(1,4)$ symmetry group. The deformed energy-momentum relation appears in a special gauge of the model. The nonlinear transformations of momenta arise from the requirement of covariance of the chosen gauge.

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I. INTRODUCTION

Various doubly special relativity (DSR) proposals have received a great amount of attention in the last years [1–18]. They have been formulated on the base of nonlinear realizations of the Lorentz group in four-dimensional space of the particle momentum [2]. It can be achieved introducing, in addition to the speed of light, one more observer independent scale¹, ζ , the latter is associated to the Planck scale (for a recent review, see [17]). In turn, the nonlinear realization implies deformed energy-momentum dispersion relation of the form

$$\eta_{\mu\nu}p^\mu p^\nu = -m^2c^2 + f(\zeta, p^0). \quad (1)$$

It is supposed that in the limit $\zeta \rightarrow 0$ one recovers the standard relation $p_\mu p^\mu = -m^2c^2$.

The attractive motivations for such kind of modification have been discussed in the literature. There is evidence on discreteness of space-time from non-perturbative quantum gravity calculations [20]. The modified energy-momentum relation implies corrections to the GZK cut-off [21], so DSR may be relevant for studying the threshold anomalies in ultra-high-energy cosmic rays [3, 22]. Astrophysical data of gamma-ray bursts can be used for bounding possible corrections to $p_\mu p^\mu = -m^2c^2$, see [14]. In the recent work [15] it was suggested that experiments with cold-atom-recoil may detect corrections to the energy-momentum relations, and $f \neq 0$ in (1) should be interpreted as a quantum gravity effect.

In this work we discuss the initial Magueijo-Smolín (MS) proposal [2], which states that all inertial observers should agree to take the deformed dispersion relation for the conserved four-momentum of a particle

$$p^2 = -m^2c^2(1 + \zeta p^0)^2. \quad (2)$$

This is invariant under the following nonlinear transformations:

$$p'^\mu = \frac{\Lambda^\mu{}_\nu p^\nu}{1 + \zeta(p^0 - \Lambda^0{}_\nu p^\nu)}. \quad (3)$$

However, the list of kinematical rules of the model is not complete, which raised a lively and controversial debate on the status of DSR [17]. One of the problems consists of the proper definition of total momentum for many particle system². Due to non-linear form of the transformations, ordinary sum of momenta does not transform as the constituents. Different covariant composition rules proposed in the literature lead to some astonishing effects, like the "soccer ball problem" and the "rainbow geometry" [8, 16].

To understand these controversial properties, it would be desirable to have in our disposal the relativistic particle model formulated in the position space, which leads to DSR relations (2), (3) in the momentum space. Despite a lot of efforts [6, 10–13], there appears to be no wholly satisfactory solution of the problem to date. It is the aim of the present work to construct the model that could be used as a laboratory for simulations of the DSR kinematics.

Nonlinear realizations of the Lorentz group on the space of physical dynamical variables often arise after fixation of a gauge in a theory with the linearly realized Lorentz group on the initial configuration space. Adopting this point of view, we study in Section 2 a singular Lagrangian on five-dimensional position space x^A ,

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¹ The idea of another invariant scale in space-time, besides c , is very old. Snyder has constructed a Lorentz invariant space-time that admits an invariant length, in one of the first attempts to avoid divergence problems [19].

² In [9] we observed that MS-type kinematics can be related with linear realization of Lorentz group in five-dimensional position space. On this base, an example of DSR model free of the problem of total momentum has been constructed.

$A = (\mu, 4)$, $\mu = 0, 1, 2, 3$, with linearly realized $SO(1, 4)$ group. To guarantee the right number of the physical degrees of freedom, we need two first-class constraints. The only $SO(1, 4)$ -invariant quadratic combinations of the variables in our disposal are p^2 , xp , x^2 . We reject x^2 as it would lead to a curved space-time³. So, we look for the model with the constraints $p^2 = 0$, $xp = 0$. They correspond to a particle with unfixed four-momentum, and without five-dimensional translation invariance. In Section 3 we show that the MS deformed energy-momentum relation arises by fixing an appropriate gauge (for one of the constraints), and the nonlinear transformation law of momenta is dictated by covariance of the gauge. Section 4 is left for conclusions.

II. $SO(1, 4)$ -INVARIANT MECHANICS

The motion of a particle in the special relativity theory can be described starting from the three-dimensional action $-mc^2 \int dt \sqrt{1 - (\frac{dx^i}{dx^0})^2}$. It implies the Hamiltonian equations

$$\frac{dx^i}{dx^0} = \frac{p^i}{\sqrt{p^2 + m^2 c^2}}, \quad \frac{dp^i}{dx^0} = 0. \quad (4)$$

The problem here is that the Lorentz transformations, $x'^\mu = \Lambda^\mu_\nu x^\nu$, act on the physical dynamical variables $x^i(x^0)$ in a higher nonlinear way. To improve this, we pass from the three-dimensional to four-dimensional formulation introducing the parametric representation $x^i(\tau)$, $x^0(\tau)$ of the particle trajectory $x^i(x^0)$. Using the relation $\frac{dx^i}{dx^0} = \frac{\dot{x}^i(\tau)}{\dot{x}^0(\tau)}$, the action acquires the form $-mc \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$. It is invariant under the local transformations which are arbitrary reparametrizations of the trajectory, $\tau \rightarrow \tau'(\tau)$. In turn, in the Hamiltonian formulation the reparametrization invariance implies the Dirac constraint which is precisely the energy-momentum relation $(p^\mu)^2 = -m^2 c^2$. Presence of the constraint becomes evident if we introduce an auxiliary variable $e(\tau)$ and rewrite the action in the equivalent form, $S = \int d\tau (\frac{1}{2e} (\dot{x}^\mu)^2 - \frac{e}{2} m^2 c^2)$. Then equation of motion for e implies the Lagrangian counterpart of the energy-momentum relation, $\frac{\delta S}{\delta e} \sim \dot{x}^2 + e^2 m^2 c^2 = 0$. Besides the constraint, the action implies the equations of motion $\dot{x}^\mu = e p^\mu$, $\dot{p}^\mu = 0$. The auxiliary variable e is not determined by these equations and enter into solution for $x^\mu(\tau)$ as an arbitrary function. The ambiguity reflects the freedom which we have in the choice of parametrization of the particle trajectory. By construction, the ambiguity is removed excluding the parameter τ from the final answers. Equivalently, we can impose a gauge

to rule out the ambiguity as well as the extra variables. The most convenient gauge is $e = 1$, $x^0 = p^0 \tau$, as it leads directly to the equations (4) for the physical variables.

In resume, to avoid a nonlinear realization of the Lorentz group in special relativity, we elevate the dimension of space from 3 to 4. In the DSR case, the Lorentz transformations are non linear in the four-dimensional space. So, by analogy with the previous case, we start from a theory with the linearly realized group in five-dimensional space. Consider the action

$$S = \int d\tau \frac{m}{2} \eta_{AB} D x^A D x^B, \quad (5)$$

where $\eta_{AB} = (-1, +1, +1, +1, +1)$, $D x^A$ stands for the "covariant derivative", $D x^A \equiv \dot{x}^A - g x^A$, and $g(\tau)$ is an auxiliary variable. The action is invariant under $SO(1, 4)$ global symmetry transformations

$$x^A \rightarrow x'^A = \Lambda^A_B x^B. \quad (6)$$

There is also the local symmetry with the parameter $\gamma(\tau)$,

$$\begin{aligned} \tau &\rightarrow \tau'(\tau); \quad \frac{d\tau'}{d\tau} = \gamma^2(\tau), \\ x^A(\tau) &\rightarrow x'^A(\tau') = \gamma(\tau) x^A(\tau), \\ g(\tau) &\rightarrow g'(\tau') = \frac{\dot{\gamma}(\tau)}{\gamma^3(\tau)} + \frac{g(\tau)}{\gamma^2(\tau)}. \end{aligned} \quad (7)$$

The transformation law for g implies a simple transformation law of the covariant derivative, $D x^A \rightarrow \frac{1}{\gamma} D x^A$. Hence g play the role of the gauge field for the symmetry.

The presence of local symmetry indicates that the model presents constraints in the Hamiltonian formulation. So we apply the Dirac method [23] to analyze the action (5). Introducing the conjugate momenta, we find the expressions

$$p_A = \frac{\partial L}{\partial \dot{x}^A} m (\dot{x}_A - g x_A), \quad p_g = \frac{\partial L}{\partial \dot{g}} = 0. \quad (8)$$

Hence there is the primary constraint, $p_g = 0$. The canonical Hamiltonian H_0 and the complete Hamiltonian H are given by the expressions

$$H_0 = \frac{1}{2m} p_A^2 + g p_A x^A, \quad H = H_0 + \lambda p_g, \quad (9)$$

where λ is the Lagrange multiplier for the primary constraint. The Poisson brackets are defined in the standard way, and equations of motion follow directly

$$\dot{x}^A = \frac{p^A}{m} + g x^A, \quad \dot{p}^A = -g p^A, \quad \dot{g} = \lambda. \quad (10)$$

From preservation in time of the primary constraint, $\dot{p}_g = 0$, we find the secondary constraint

$$p_A x^A = 0. \quad (11)$$

³ There are proposals considering the de Sitter as the underlying space for DSR theories [4–6, 18].

In turn, it implies the tertiary constraint

$$p_A^2 = 0. \quad (12)$$

The Dirac procedure stops on this stage, all the constraints obtained belong to the first class.

Since we deal with a constrained theory, our first task is to specify the physical-sector variables [26]. The initial phase space is parameterized by 12 variables x^A , p^B , g , p_g . Taking into account that each first-class constraint rules out two variables, the number of phase-space physical variables is $12 - 2 \times 3 = 6$, as it should be for DSR-particle. We note that Eq. (10) does not determine the Lagrange multiplier λ , which enters as an arbitrary function into solutions to the equations of motion. According to the general theory [23–25], variables with ambiguous dynamics do not represent the observable quantities. For our case, all the initial variables turn out to be ambiguous.

To construct the unambiguous variables, we note that the quantities $\pi^\mu = \frac{p^\mu}{p^4}$, $y^\mu = \frac{x^\mu}{x^4}$, obey $\dot{\pi}^\mu = 0$, $\dot{y}^\mu = \frac{e}{m}(\pi^\mu - y^\mu)$, where $e \equiv \frac{p^4}{x^4}$. Since these equations resemble those for a spinless relativistic particle, the remaining ambiguity due to e has the well-known interpretation, being related with reparametrization invariance of the theory. In accordance with this, we can assume that $y^\mu(\tau)$ represent the parametric equations of the trajectory $y^i(t)$. The reparametrization-invariant variable $y^i(t)$ has deterministic evolution $\frac{dy^i}{dt} = c \frac{\pi^i - y^i}{\pi^0 - y^0}$.

We can also look for the gauge-invariant combinations on the phase-space. The well known remarkable property of the Hamiltonian formalism is that there are the phase-space coordinates for which the Hamiltonian vanishes [25]. In these coordinates trajectories look like the straight lines. For the case, the unambiguous variables with this property are π^μ , $\hat{x}^\mu \equiv y^\mu - \pi^\mu$.

III. DSR GAUGE

In this Section we reproduce the MS DSR kinematics starting from the $SO(1,4)$ model. First, we obtain the MS dispersion relation (2) imposing a particular gauge in our model. Generally, neither the global nor the local symmetries survive separately in the gauge fixed version. But we can look for their combination that does not spoil the gauge condition. Following this line, we arrive at the MS transformation law of the momenta (3).

According the Dirac algorithm, each first class constraint must be accompanied by some gauge condition of the form $h(x, p) = 0$, where the function h must be chosen such that the system formed by constraints and gauges is second class. The constraints and the gauges then can be used to represent part of the phase space variables through other. Equations of motion for the remaining variables are obtained by substituting the constraints and gauges into the equations already found.

Let us choose the gauge $g = 0$ for the constraint $p_g = 0$. This gauge fixes the local symmetry, as it should be,

$$g' = \frac{\dot{\gamma}}{\gamma^3} + \frac{g}{\gamma^2} \Big|_{g=0} \Rightarrow \dot{\gamma} = 0. \quad (13)$$

We are, then, left with two constraints. To obtain a deformed dispersion relation, we impose the gauge $p^4 = mch(\zeta, p^0)$ for the constraint $p_A x^A = 0$. Using this expression in the constraint (12), we obtain

$$p_\mu p^\mu = -m^2 c^2 h^2(\zeta, p^0). \quad (14)$$

We wrote the function h depending on the arguments ζ and p^0 but one is free to choose the particular dispersion relation he wants. We point out that the scale ζ is gauge-noninvariant notion in this model ⁴.

We now turn to the induced nonlinear Lorentz transformation of momenta. Under the symmetries (6), (7), the conjugated momentum $p^A = mDx^A$ transforms as

$$p^A \rightarrow p'^A = \frac{1}{\gamma} \Lambda^A_B p^B. \quad (15)$$

For $SO(1,3)$ -subgroup ⁵

$$\Lambda^A_B = \begin{pmatrix} \Lambda^\mu_\nu & 0 \\ 0 & 1 \end{pmatrix}, \quad (16)$$

we have

$$p^\mu \rightarrow p'^\mu = \frac{1}{\gamma} \Lambda^\mu_\nu p^\nu, \quad p^4 \rightarrow p'^4 = \frac{1}{\gamma} \Lambda^4_A p^A = \frac{1}{\gamma} p^4. \quad (17)$$

Now, as it often happens in gauge theories, global symmetry of the gauge-fixed formulation is a combination of the initial global symmetry and local symmetry with specially chosen parameter γ . Since the gauge $p^4 = mch(\zeta, p^0)$ is not preserved by the transformations (6) and (7) separately, one is forced to search for their combination, (17), which preserves the gauge. Imposing the covariance of the gauge

$$p^4 = mch(\zeta, p^0) \Leftrightarrow p'^4 = mch(\zeta, p'^0), \quad (18)$$

we obtain the equation for determining γ

$$h(\zeta, p^0) = \gamma h(\zeta, \frac{1}{\gamma} \Lambda^0_\mu p^\mu). \quad (19)$$

In the gauge $g = 0$, we have $p_A = \text{const.}$ on-shell, so Eq. (19) is consistent with (13). Eqs. (17) with this

⁴ It is worth noting that a gauge-fixed formulation, considered irrespectively to the initial one, generally has the physical sector different from those of the initial theory [24].

⁵ We discuss only the induced Lorentz transformations. The remaining transformations are boosts in the fifth dimension. In the gauge-fixed formulation they produce the nonlinear transformations which play the role of four-dimensional translations.

γ provides a non linear realization of the Lorentz group which leaves invariant the deformed energy-momentum relation (14).

Let us specify all this for MS model. If we fix the gauge $p^4 = mc(1 + \zeta p^0)$, the constraint $p_A^2 = 0$ acquires the form of MS dispersion relation (2). Enforcing covariance of the gauge, the equation (19) for determining γ reads

$$1 + \zeta p^0 = \gamma(1 + \zeta \frac{1}{\gamma} \Lambda^0_{\mu} p^{\mu}). \quad (20)$$

So, γ is given by

$$\gamma = 1 + \zeta(p^0 - \Lambda^0_{\mu} p^{\mu}). \quad (21)$$

Using this γ in Eq. (17), we see that the momenta p^{μ} transform according to Eq. (3).

IV. CONCLUDING REMARKS

We have constructed an example of the relativistic particle model (5) on five-dimensional flat space-time with linearly realized $SO(1, 4)$ group of global symmetries and without the five-dimensional translation invariance. Due to the local symmetry presented in the action, the number of physical degrees of freedom of the model is the same as for the particle of special relativity theory. We have applied the model to simulate kinematics of the Magueijo-Smolín doubly-special-relativity proposal. It was done by an appropriate fixation of a gauge for the constraint (11), that leads to the MS deformed dispersion relation (2). The nonlinear transformation law of momenta (3) was found from the requirement of covariance of the gauge-fixed version.

We finish with the comment on a transformation law for the spatial coordinates. Using the parameter γ obtained in Eq. (21), the transformation of the configuration-space coordinates can be found from (6) and (7)

$$x^{\mu} \rightarrow x'^{\mu} = [1 + \zeta(p^0 - \Lambda^0_{\mu} p^{\mu})] \Lambda^{\mu}_{\nu} x^{\nu}, \quad (22)$$

$$x^4 \rightarrow x'^4 = [1 + \zeta(p^0 - \Lambda^0_{\mu} p^{\mu})] x^4. \quad (23)$$

The component x^4 is affected only by a scale factor. The coordinates x^{μ} transform as usually happens in DSR theories: we have a transformation law that is energy-momentum dependent. These transformations were obtained in the work [7] from the requirement that the free field defined on DSR space (2) should have plane-wave solutions of the form $\phi \sim Ae^{-ip_{\mu}x^{\mu}}$, then the contraction $p_{\mu}x^{\mu}$ must remain linear in any frame. We point out that it turns out to be true in our model

$$\eta_{\mu\nu} p'^{\mu} x'^{\nu} = \eta_{\mu\nu} (\frac{1}{\gamma} \Lambda^{\mu}_{\alpha} p^{\alpha}) (\gamma \Lambda^{\nu}_{\beta} x^{\beta}) = \eta_{\alpha\beta} p^{\alpha} x^{\beta}. \quad (24)$$

Eq. (22) leads also to the energy-dependent metric of the position space [8]. There are some attempts to interpret p^0 in this case, see [7, 8].

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